AP CALCULUS AB AND BC

UNIT 9 BC ONLY

Parametric Equations, Polar Coordinates, and Vector-Valued Functions

AP EXAM WEIGHTING **11–12%** BC ONLY

CLASS PERIODS **~10–11**

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 9

Multiple-choice: ~25 questions
Free-response: 3 questions

UNIT 9

AP EXAM WEIGHTING	~11–12% BC
CLASS PERIODS	~10–11 BC

Parametric Equations, Polar Coordinates, and Vector-Valued Functions

BIG IDEA 1
Change **CHA**
- How can we model motion not constrained to a linear path?

BIG IDEA 3
Analysis of Functions **FUN**
- How does the chain rule help us to analyze graphs defined using parametric equations or polar functions?

Developing Understanding

In this unit, students will build on their understanding of straight-line motion to solve problems in which particles are moving along curves in the plane. Students will define parametric equations and vector-valued functions to describe planar motion and apply calculus to solve motion problems. Students will learn that polar equations are a special case of parametric equations and will apply calculus to analyze graphs and determine lengths and areas. This unit should be treated as an opportunity to reinforce past learning and transfer knowledge and skills to new situations, rather than as a new list of facts or strategies to memorize.

Building the Mathematical Practices

1.D 1.E 2.D 3.D

As students transition to parametric and vector-valued functions, they'll need to practice previously learned concepts and skills to reinforce the new procedures and representations they're learning in Unit 9. As with particle motion on a line, students learning to handle motion in the plane will need to practice interpreting which procedure is needed for different scenarios (differentiation or integration) and solving for speed, velocity, distance traveled, or initial position.

Reinforce the importance of precise notation, particularly regarding the variable of differentiation, as well as correct application of the chain rule. Leibniz notation helps students to remember how to find the derivative of y with respect to x for coordinates defined using the parameter t:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt} \cdot \frac{dt}{dx}, \text{ provided } \frac{dx}{dt} \neq 0.$$

Since $\frac{dy}{dx}$ is in terms of t, students must be particularly careful when determining $\frac{d^2y}{dx^2}$.

Similarly, using definite integrals to represent lengths and areas defined by polar curves is based on the same principles as calculating lengths and areas defined by the graphs of more familiar functions (i.e., the limit of a Riemann sum). Students will need to practice with trigonometric identities, radian measures and formulas for arc length and area of a sector to reinforce practice with associated calculus topics.

Preparing for the AP Exam

While students need more experience shifting mindsets from rectangular to polar coordinate systems, errors in arithmetic, algebra, trigonometry, and procedures such as the chain rule are often even more problematic. Provide opportunities for students to reinforce familiar skills and concepts as they practice new techniques in preparation for the AP Exam. As with analysis of graphs, sign charts can be useful tools for identifying answers to questions about the direction of motion or whether speed is increasing or decreasing, for example. To earn points for justification, however, students must connect their work to a relevant definition or theorem, as in the **Scoring Guidelines for 2017 AB5**. Continue to emphasize accounting for initial values, as in past units, as well as precise communication and notational fluency. Paying attention to subscripts in problems involving more than one particle is essential to clear communication.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~10–11 CLASS PERIODS
CHA-3	**9.1 Defining and Differentiating Parametric Equations**	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
	9.2 Second Derivatives of Parametric Equations	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
CHA-6	**9.3 Finding Arc Lengths of Curves Given by Parametric Equations**	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
CHA-3	**9.4 Defining and Differentiating Vector-Valued Functions**	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
FUN-8	**9.5 Integrating Vector-Valued Functions**	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	9.6 Solving Motion Problems Using Parametric and Vector-Valued Functions	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-3	**9.7 Defining Polar Coordinates and Differentiating in Polar Form**	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
CHA-5	**9.8 Find the Area of a Polar Region or the Area Bounded by a Single Polar Curve**	3.D Apply an appropriate mathematical definition, theorem, or test.	
	9.9 Finding the Area of the Region Bounded by Two Polar Curves	3.D Apply an appropriate mathematical definition, theorem, or test.	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 9. Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	9.2	**Numbered Heads Together** Organize the class into groups of four and give each student a number. Give the class one set of parametric equations and have students individually find the second derivative. When all students in a group have finished, have them stand up to discuss their answers, make any necessary corrections, and then sit back down. Choose a group and call a specific student number so that the student can share the answer with the class.
2	9.5 9.6	**Scavenger Hunt** Create cards containing a parametric initial value question and the answer to a different question. Students can start at any question, moving around the room to make a full circuit of all questions once complete. Focus on giving students an initial value and asking for one or both component values at a different t- or x-value (e.g., $x'(t) = 2t$, $y'(t) = \cos(t)$, and position is $(2, 0)$ at $t = \pi$. Find position at $t = \frac{5\pi}{6}$.).
3	9.8	**Create Representations** Give students polar equations of various types: circle, limaçon with inner loop, cardioid, dimpled limaçon, rose curve, or lemniscates. Have students create a table of values, sketch the graph of the curve using rectangular coordinates, and sketch the graph of the curve using polar coordinates. Preface this activity by modeling the steps with one function on large paper, using wiki sticks to show the y-values as heights becoming the r-values as radii.
4	9.8	**Paraphrasing** Give students a proof which derives the polar area formula. Ask them to restate the meaning and derivation of this formula in their own words. Have them also compare and contrast this formula to the integration used to find areas under functions in the Cartesian coordinate system.
5	9.9	**Stand Up, Hand Up, Pair Up** Give each pair of students a polar area question to solve. Once they have completed both roles obtaining only the integral setup, have them use a calculator to find the numeric solution and confirm with you before standing up to switch pairs.

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

AVAILABLE RESOURCES

- Classroom Resource > **Vectors**
- External Resource > **Davidson Next**

TOPIC 9.1

Defining and Differentiating Parametric Equations

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.G

Calculate derivatives of parametric functions.

BC ONLY

ESSENTIAL KNOWLEDGE

CHA-3.G.1

Methods for calculating derivatives of real-valued functions can be extended to parametric functions. **BC ONLY**

CHA-3.G.2

For a curve defined parametrically, the value of $\frac{dy}{dx}$ at a point on the curve is the slope of the line tangent to the curve at that point. $\frac{dy}{dx}$, the slope of the line tangent to a curve defined using parametric equations, can be determined by dividing $\frac{dy}{dt}$ by $\frac{dx}{dt}$, provided $\frac{dx}{dt}$ does not equal zero. **BC ONLY**

TOPIC 9.2

Second Derivatives of Parametric Equations

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.G

Calculate derivatives of parametric functions.

BC ONLY

ESSENTIAL KNOWLEDGE

CHA-3.G.3

$\frac{d^2y}{dx^2}$ can be calculated by dividing $\frac{d}{dt}\left(\frac{dy}{dx}\right)$ by $\frac{dx}{dt}$. **BC ONLY**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Classroom Resource > **Vectors**
- External Resource > **Davidson Next**

SUGGESTED SKILL

Implementing Mathematical Processes

1.D
Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

AVAILABLE RESOURCES

- Classroom Resource > **Vectors**
- External Resource > **Davidson Next**

TOPIC 9.3

Finding Arc Lengths of Curves Given by Parametric Equations

Required Course Content

ENDURING UNDERSTANDING

CHA-6

Definite integrals allow us to solve problems involving the accumulation of change in length over an interval.

LEARNING OBJECTIVE

CHA-6.B

Determine the length of a curve in the plane defined by parametric functions, using a definite integral.

BC ONLY

ESSENTIAL KNOWLEDGE

CHA-6.B.1

The length of a parametrically defined curve can be calculated using a definite integral.

BC ONLY

TOPIC 9.4

Defining and Differentiating Vector-Valued Functions

SUGGESTED SKILL

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

AVAILABLE RESOURCES

- Classroom Resource > **Vectors**
- External Resource > **Davidson Next**

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.H

Calculate derivatives of vector-valued functions. **BC ONLY**

ESSENTIAL KNOWLEDGE

CHA-3.H.1

Methods for calculating derivatives of real-valued functions can be extended to vector-valued functions. **BC ONLY**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Classroom Resource > **Vectors**
- External Resource > **Davidson Next**

TOPIC 9.5

Integrating Vector-Valued Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-8

Solving an initial value problem allows us to determine an expression for the position of a particle moving in the plane.

LEARNING OBJECTIVE

FUN-8.A

Determine a particular solution given a rate vector and initial conditions.

BC ONLY

ESSENTIAL KNOWLEDGE

FUN-8.A.1

Methods for calculating integrals of real-valued functions can be extended to parametric or vector-valued functions. **BC ONLY**

TOPIC 9.6

Solving Motion Problems Using Parametric and Vector-Valued Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-8

Solving an initial value problem allows us to determine an expression for the position of a particle moving in the plane.

LEARNING OBJECTIVE

FUN-8.B

Determine values for positions and rates of change in problems involving planar motion. **BC ONLY**

ESSENTIAL KNOWLEDGE

FUN-8.B.1

Derivatives can be used to determine velocity, speed, and acceleration for a particle moving along a curve in the plane defined using parametric or vector-valued functions.
BC ONLY

FUN-8.B.2

For a particle in planar motion over an interval of time, the definite integral of the velocity vector represents the particle's displacement (net change in position) over the interval of time, from which we might determine its position. The definite integral of speed represents the particle's total distance traveled over the interval of time. **BC ONLY**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Classroom Resource > **Vectors**
- External Resource > **Davidson Next**

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

AVAILABLE RESOURCE

- External Resource > **Davidson Next**

TOPIC 9.7

Defining Polar Coordinates and Differentiating in Polar Form

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.G

Calculate derivatives of functions written in polar coordinates. **BC ONLY**

ESSENTIAL KNOWLEDGE

FUN-3.G.1

Methods for calculating derivatives of real-valued functions can be extended to functions in polar coordinates. **BC ONLY**

FUN-3.G.2

For a curve given by a polar equation $r = f(\theta)$, derivatives of r, x, and y with respect to θ, and first and second derivatives of y with respect to x can provide information about the curve. **BC ONLY**

TOPIC 9.8
Find the Area of a Polar Region or the Area Bounded by a Single Polar Curve

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.D

Calculate areas of regions defined by polar curves using definite integrals. **BC ONLY**

ESSENTIAL KNOWLEDGE

CHA-5.D.1

The concept of calculating areas in rectangular coordinates can be extended to polar coordinates. **BC ONLY**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCE

- External Resource > **Davidson Next**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCE

- External Resource > **Davidson Next**

TOPIC 9.9

Finding the Area of the Region Bounded by Two Polar Curves

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.D

Calculate areas of regions defined by polar curves using definite integrals. **BC ONLY**

ESSENTIAL KNOWLEDGE

CHA-5.D.2

Areas of regions bounded by polar curves can be calculated with definite integrals. **BC ONLY**