AP CALCULUS AB AND BC

UNIT 10 BC ONLY

Infinite Sequences and Series

AP	AP EXAM WEIGHTING	17–18% BC ONLY
	CLASS PERIODS	~17–18

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 10

Multiple-choice: ~45 questions
Free-response: 3 questions

UNIT 10

AP EXAM WEIGHTING 17–18% BC

CLASS PERIODS ~17–18 BC

Infinite Sequences and Series

BIG IDEA 1
Limits LIM

- How can the sum of infinitely many discrete terms be a finite value or represent a continuous function?

Developing Understanding

In this unit, students need to understand that a sum of infinitely many terms may converge to a finite value. They can develop intuition by exploring graphs, tables, and symbolic expressions for series that converge and diverge and for Taylor polynomials. Students should build connections to past learning, such as how evaluating improper integrals relates to the integral test or how using limiting cases of power series to represent continuous functions relates to differentiation and integration. Students who rely solely on memorizing a long list of tests and procedures typically find little success achieving a lasting conceptual understanding of series.

Building the Mathematical Practices

1.E 1.F 2.C 3.D

In Unit 10, students will need to develop proficiency with complex series notation and the ability to communicate their reasoning. Emphasize appropriate use of notation, precision of language, and establishing conditions for using a particular test. Remind students that a sound justification relies upon both mathematical evidence and reasons why that evidence supports the conclusion.

Additionally, students will need to practice determining which application is appropriate for different scenarios (for example, using the definitions of harmonic or p-series to classify certain infinite series) and then applying associated procedures accurately. Students will also need to practice using Taylor polynomials to approximate the value of a function, choosing and implementing an appropriate method to bound the error involved in the approximation, and effectively communicating supporting work.

Connecting representations is an important skill to develop in this unit. For example, students will need to identify infinite power series to represent functions presented symbolically or move between graphic and symbolic representations of an interval of convergence.

Preparing for the AP Exam

Students are more likely to demonstrate an incomplete understanding of series or to struggle with communicating their understanding of it compared to other topics. Continue to model and expect correct notation and language to present solutions, explain reasoning, and justify conclusions. For example, using the ratio test to find a radius of convergence, or operating on a known series to create another series, requires proficient, well-presented algebra. Applying a convergence test requires explicit verification that all necessary conditions are met. Determining that a given number is an error bound requires calculating an appropriate value and communicating that the value is less than the given number.

Intentional focus on the recurrent theme of using limiting cases to move from discrete approximations to analytic calculations and determinations is one way to help students to finish the year with a strong performance on the AP Exam and to come away with an enduring, meaningful understanding of calculus.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~17–18 CLASS PERIODS
LIM-7	**10.1** Defining Convergent and Divergent Infinite Series	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.2 Working with Geometric Series	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.3 The *n*th Term Test for Divergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.4 Integral Test for Convergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.5 Harmonic Series and *p*-Series	3.B Identify an appropriate mathematical definition, theorem, or test to apply.	
	10.6 Comparison Tests for Convergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.7 Alternating Series Test for Convergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.8 Ratio Test for Convergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.9 Determining Absolute or Conditional Convergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.10 Alternating Series Error Bound	1.E Apply appropriate mathematical rules or procedures, with and without technology.	

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UNIT AT A GLANCE *(cont'd)*

Enduring Understanding	Topic	Suggested Skills	Class Periods ~17–18 CLASS PERIODS
LIM-8	**10.11 Finding Taylor Polynomial Approximations of Functions**	3.D Apply an appropriate mathematical definition, theorem, or test. 2.C Identify a re-expression of mathematical information presented in a given representation.	
	10.12 Lagrange Error Bound	1.F Explain how an approximated value relates to the actual value.	
	10.13 Radius and Interval of Convergence of Power Series	2.C Identify a re-expression of mathematical information presented in a given representation.	
	10.14 Finding Taylor or Maclaurin Series for a Function	2.C Identify a re-expression of mathematical information presented in a given representation.	
	10.15 Representing Functions as Power Series	3.D Apply an appropriate mathematical definition, theorem, or test.	
	Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 10. Review the results in class to identify and address any student misunderstandings.		

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	10.1	**Predict and Confirm** Demonstrate a geometric series, the harmonic series, and the alternating series by distributing pieces of a donut, pizza, or licorice. Ask the class to predict how much the student(s) will eventually receive in total. For instance, give three students each one-fourth, then one-fourth of the remaining fourth, and so on. Students should guess that since the remaining part is approaching zero, each student will eventually receive one-third. For alternating harmonic series, give one student a whole piece, then take away ½, then give ⅓, then take away ¼, and so forth.
2	10.2 10.3 10.4 10.5 10.6 10.7 10.8	**Graphic Organizer** Put students in groups with poster paper and have them organize and explain all the series tests using pictures, text, flowcharts, cartoons, or other drawings. Have them include each test's conditions and how to choose which test to apply.
3	10.13	**Odd One Out** Begin by modeling an example, such as three images that are pieces of furniture and one image of a houseplant, explaining that the houseplant is the "odd one out" because it's not like the other images. Then distribute a set of series such that all of the series, except one, have something in common. For example, all of the series except one could have the same type of interval of convergence (open, closed, open at left or right endpoint). Or they could converge only at the series' center or they could converge for all real numbers. Then have students decide which series in the set is the "odd one out."
4	10.15	**Collaborative Poster** Use this strategy for groups to create their own free-response questions. Give each group of four students a basic series, such as $\sin(x)$, $\cos(x)$, e^x, or $\frac{1}{1+x}$. Ask the first two members to choose a manipulation of the series and show work to complete the task. Ask the final two members to watch silently and confirm the first two steps. Then the final two members choose further actions to perform on the new series (i.e., differentiate, integrate, and find interval of convergence).

TOPIC 10.1
Defining Convergent and Divergent Infinite Series

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-7.A.1

The nth partial sum is defined as the sum of the first n terms of a series. **BC ONLY**

LIM-7.A.2

An infinite series of numbers converges to a real number S (or has sum S), if and only if the limit of its sequence of partial sums exists and equals S. **BC ONLY**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

TOPIC 10.2
Working with Geometric Series

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-7.A.3

A geometric series is a series with a constant ratio between successive terms. **BC ONLY**

LIM-7.A.4

If a is a real number and r is a real number such that $|r| < 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. **BC ONLY**

TOPIC 10.3
The *n*th Term Test for Divergence

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-7.A.5

The *n*th term test is a test for divergence of a series. **BC ONLY**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

TOPIC 10.4
Integral Test for Convergence

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges.

BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.6

The integral test is a method to determine whether a series converges or diverges.

BC ONLY

TOPIC 10.5
Harmonic Series and p-Series

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges.

BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.7

In addition to geometric series, common series of numbers include the harmonic series, the alternating harmonic series, and p-series.

BC ONLY

SUGGESTED SKILL

Justification

3.B

Identify an appropriate mathematical definition, theorem, or test to apply.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

TOPIC 10.6
Comparison Tests for Convergence

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-7.A.8

The comparison test is a method to determine whether a series converges or diverges. **BC ONLY**

LIM-7.A.9

The limit comparison test is a method to determine whether a series converges or diverges. **BC ONLY**

TOPIC 10.7
Alternating Series Test for Convergence

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-7.A.10

The alternating series test is a method to determine whether an alternating series converges. **BC ONLY**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- The Exam > **Commentary on the Instructions for the Free Response Section of the AP Calculus Exams**
- External Resource > **Davidson Next**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

TOPIC 10.8
Ratio Test for Convergence

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-7.A.11

The ratio test is a method to determine whether a series of numbers converges or diverges. **BC ONLY**

EXCLUSION STATEMENT

The nth term test for divergence, and the integral test, comparison test, limit comparison test, alternating series test, and ratio test for convergence are assessed on the AP Calculus BC Exam. Other methods are not assessed on the exam. However, teachers may include additional methods in the course, if time permits.

TOPIC 10.9
Determining Absolute or Conditional Convergence

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges.

BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.12

A series may be absolutely convergent, conditionally convergent, or divergent.

BC ONLY

LIM-7.A.13

If a series converges absolutely, then it converges. **BC ONLY**

LIM-7.A.14

If a series converges absolutely, then any series obtained from it by regrouping or rearranging the terms has the same value.

BC ONLY

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

TOPIC 10.10
Alternating Series Error Bound

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.B

Approximate the sum of a series. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-7.B.1

If an alternating series converges by the alternating series test, then the alternating series error bound can be used to bound how far a partial sum is from the value of the infinite series. **BC ONLY**

TOPIC 10.11
Finding Taylor Polynomial Approximations of Functions

Required Course Content

ENDURING UNDERSTANDING

LIM-8

Power series allow us to represent associated functions on an appropriate interval.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
LIM-8.A Represent a function at a point as a Taylor polynomial. **BC ONLY**	**LIM-8.A.1** The coefficient of the nth degree term in a Taylor polynomial for a function f centered at $x = a$ is $\frac{f^{(n)}(a)}{n!}$. **BC ONLY**
	LIM-8.A.2 In many cases, as the degree of a Taylor polynomial increases, the nth degree polynomial will approach the original function over some interval. **BC ONLY**
LIM-8.B Approximate function values using a Taylor polynomial. **BC ONLY**	**LIM-8.B.1** Taylor polynomials for a function f centered at $x = a$ can be used to approximate function values of f near $x = a$. **BC ONLY**

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

SUGGESTED SKILL

Implementing Mathematical Processes

1.F

Explain how an approximated value relates to the actual value.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- Classroom Resource > **Approximation**
- External Resource > **Davidson Next**

TOPIC 10.12
Lagrange Error Bound

Required Course Content

ENDURING UNDERSTANDING

LIM-8

Power series allow us to represent associated functions on an appropriate interval.

LEARNING OBJECTIVE

LIM-8.C

Determine the error bound associated with a Taylor polynomial approximation. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-8.C.1

The Lagrange error bound can be used to determine a maximum interval for the error of a Taylor polynomial approximation to a function. **BC ONLY**

LIM-8.C.2

In some situations, the alternating series error bound can be used to bound the error of a Taylor polynomial approximation to the value of a function. **BC ONLY**

TOPIC 10.13
Radius and Interval of Convergence of Power Series

Required Course Content

ENDURING UNDERSTANDING

LIM-8

Power series allow us to represent associated functions on an appropriate interval.

LEARNING OBJECTIVE

LIM-8.D

Determine the radius of convergence and interval of convergence for a power series. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-8.D.1

A power series is a series of the form $\sum_{n=0}^{\infty} a_n(x-r)$, where n is a non-negative integer, $\{a_n\}$ is a sequence of real numbers, and r is a real number. **BC ONLY**

LIM-8.D.2

If a power series converges, it either converges at a single point or has an interval of convergence. **BC ONLY**

LIM-8.D.3

The ratio test can be used to determine the radius of convergence of a power series. **BC ONLY**

LIM-8.D.4

The radius of convergence of a power series can be used to identify an open interval on which the series converges, but it is necessary to test both endpoints of the interval to determine the interval of convergence. **BC ONLY**

LIM-8.D.5

If a power series has a positive radius of convergence, then the power series is the Taylor series of the function to which it converges over the open interval. **BC ONLY**

LIM-8.D.6

The radius of convergence of a power series obtained by term-by-term differentiation or term-by-term integration is the same as the radius of convergence of the original power series. **BC ONLY**

SUGGESTED SKILL

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

SUGGESTED SKILL

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- AP Online Teacher Community Discussion > **Question on Taylor Polynomials**
- External Resource > **Davidson Next**

TOPIC 10.14

Finding Taylor or Maclaurin Series for a Function

Required Course Content

ENDURING UNDERSTANDING

LIM-8

Power series allow us to represent associated functions on an appropriate interval.

LEARNING OBJECTIVE

LIM-8.E

Represent a function as a Taylor series or a Maclaurin series. **BC ONLY**

LIM-8.F

Interpret Taylor series and Maclaurin series. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-8.E.1

A Taylor polynomial for $f(x)$ is a partial sum of the Taylor series for $f(x)$. **BC ONLY**

LIM-8.F.1

The Maclaurin series for $\frac{1}{1-x}$ is a geometric series. **BC ONLY**

LIM-8.F.2

The Maclaurin series for $\sin x$, $\cos x$, and e^x provides the foundation for constructing the Maclaurin series for other functions. **BC ONLY**

TOPIC 10.15
Representing Functions as Power Series

Required Course Content

ENDURING UNDERSTANDING

LIM-8

Power series allow us to represent associated functions on an appropriate interval.

LEARNING OBJECTIVE

LIM-8.G

Represent a given function as a power series. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-8.G.1

Using a known series, a power series for a given function can be derived using operations such as term-by-term differentiation or term-by-term integration, and by various methods (e.g., algebraic processes, substitutions, or using properties of geometric series). **BC ONLY**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**