Name________________________________

Embedded Section : Series

Certain functions can be approximated by a polynomial in the form of an infinite series. The following series are approximations for the functions when the x-values are close to 0.

$e^x =$

$\sin x =$

$\cos x =$

$\dfrac{1}{1-x} =$

$\dfrac{1}{1+x} =$

Why is this necessary??

First, differentiation and integration of power series can be performed term by term and is hence particularly easy. Second , the (truncated) series can be used to compute function values approximately. Finally, algebraic operations can often be done much more readily on the power series representation.

Right now, since we are just in the very beginning stages of series approximations, we will be doing minimal operations with them.

Examples: Find the derivatives of the following functions using their polynomial approximations.

$$\frac{d}{dx}\left(e^x\right)=$$

$$\frac{d}{dx}(\sin x)=$$

$$\frac{d}{dx}(\cos x)=$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right)=$$

$$\frac{d}{dx}\left(\frac{1}{1+x}\right)=$$

We can also use the polynomial approximation to find values of functions in a certain neighborhood. In our case, all of our approximations are for values close to 0, so we can only approximate for values in the interval [-1, 1].

Also, in order to approximate a value for a function, we need to know what order of the polynomial to use. For example, If I asked you to find the 4^{th} order approximation for cos 1, you would write the 4^{th} order series for cos x, replacing 1 for x and then solving:

$$\cos 1 = 1 - \frac{1^2}{2!} + \frac{1^4}{4!} = \frac{13}{24}$$

Note that 4^{th} order does not mean four terms, but rather you will stop when your function has a 4^{th} power in it.

1.) Find the 7^{th} order approximation of $\sin \frac{1}{2}$.

2.) Find the third order approximation of $e^{.8}$.

Name______________________________

Series Worksheet

Find the following derivatives or integrals using polynomial (series) approximations using the first three terms.

1.) $\int -2\sin 2x\,dx \approx$

2.) $\frac{d}{dx}\left(3e^{-x}\right) \approx$

3.) $\int \frac{1}{1+2x}dx \approx$

4.) $\frac{d}{dx}(\cos 5x) \approx$

Name________________________________

Embedded Section : Series

Certain functions can be approximated by a polynomial in the form of an infinite series. The following series are approximations for the functions when the x-values are close to 0.

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{(n)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$$

<u>Why is this necessary??</u>

First, differentiation and integration of power series can be performed term by term and is hence particularly easy. Second , the (truncated) series can be used to compute function values approximately. Finally, algebraic operations can often be done much more readily on the power series representation.

Right now, since we are just in the very beginning stages of series approximations, we will be doing minimal operations with them.

Examples: Find the derivatives of the following functions using their polynomial approximations.

$$\frac{d}{dx}(e^x) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\right)$$

$$= 1 + x + \frac{x^2}{2!} + \cdots\cdots = e^x$$ ✓

$$\frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= 1 - \frac{\cancel{3}x^2}{\cancel{3!}\,2!} + \frac{\cancel{5}x^4}{\cancel{5!}\,4!} - \frac{\cancel{7}x^6}{\cancel{7!}\,6!} + \cdots\cdots = \cos x$$ ✓

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots\right)$$

$$= -\frac{\cancel{2}x}{\cancel{2!}} + \frac{\cancel{4}x^3}{\cancel{4!}\,3!} - \frac{\cancel{6}x^5}{\cancel{6!}\,5!} + \cdots = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots = -\sin x$$ ✓

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots\right)$$

$$= 1 + 2x + 3x^2 + \cdots\cdots$$

$$\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{d}{dx}\left(1 - x + x^2 - x^3 + \cdots\right)$$

$$= -1 + 2x - 3x^2 + \cdots\cdots$$

We can also use the polynomial approximation to find values of functions in a certain neighborhood. In our case, all of our approximations are for values close to 0, so we can only approximate for values in the interval [-1, 1].

Also, in order to approximate a value for a function, we need to know what order of the polynomial to use. For example, If I asked you to find the 4th order approximation for cos 1, you would write the 4th order series for cos x, replacing 1 for x and then solving:

$$\cos 1 = 1 - \frac{1^2}{2} + \frac{1^4}{4!} = \frac{13}{24} = .5417 \qquad \cos 1 = .5403$$

Note that 4th order does not mean four terms, but rather you will stop when your function has a 4th power in it.

1.) Find the 7th order approximation of $\sin\frac{1}{2}$.

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sin(1/2) \approx 1/2 - \frac{(1/2)^3}{3!} + \frac{(1/2)^5}{5!} - \frac{(1/2)^7}{7!} = .4794255332$$

2.) Find the third order approximation of $e^{.8}$. actual $\sin(\frac{1}{2}) = .4794255386$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$e^{.8} \approx 1 + .8 + \frac{(.8)^2}{2!} + \frac{(.8)^3}{3!} = 2.20533\overline{3}$$

actual $e^{.8} = 2.22554$

Name________________________________

Series Worksheet

Find the following derivatives or integrals using polynomial (series) approximations using the first three terms.

1.) $\int -2\sin 2x\,dx \approx -2\int\left[(2x)-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\frac{(2x)^7}{7!}+\dots\right]dx$

$-2\int\left(2x-\frac{8x^3}{3!}+\frac{32x^5}{5!}-\frac{128x^7}{7!}+\dots\right)dx$

$-2\left[x^2-\frac{8x^4}{4!}+\frac{32x^6}{6!}-\frac{64x^8}{8!}+\dots\right]$

2.) $\frac{d}{dx}\left(3e^{-x}\right)\approx \frac{d}{dx}\left[3\left(1+(-x)+\frac{(-x)^2}{2!}+\frac{(-x)^3}{3!}+\dots\right)\right]$ ~~dx~~

$\frac{d}{dx}\left[3\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\dots\right)\right]dx$

check $-3e^{-x}$

$=3\left[-1+\frac{2x}{2!}-\frac{3x^2}{3!}+\dots\right]$

$=3\left[-1+x-\frac{x^2}{2!}+\dots\right]=-3\left[1-x+\frac{x^2}{2!}+\dots\right]=-3e^{-x}$ ✓

3.) $\int\frac{1}{1+2x}dx\approx\int\left(1-(2x)+(2x)^2-(2x)^3+(2x)^4+\dots\right)dx=\int 1-2x+4x^2-8x^3+\dots$

$\frac{1}{1-(-2x)}$

$=x-\frac{2x^2}{2}+\frac{4x^3}{3}-\frac{8x^4}{4}+\dots+C$

$=x-x^2+\frac{4}{3}x^3-2x^4+\dots+C$

check $-5\sin 5x$

4.) $\frac{d}{dx}(\cos 5x)\approx\frac{d}{dx}\left[1-\frac{(5x)^2}{2!}+\frac{(5x)^4}{4!}-\frac{(5x)^6}{6!}+\dots\right]$

$=\frac{d}{dx}\left[1-\frac{25x^2}{2!}+\frac{625x^4}{4!}-\frac{5^6x^6}{6!}+\dots\right]$

$=-\frac{50x}{2!}+\frac{625\cdot 4x^3}{4!}-\frac{6\cdot 5^6x^5}{6!}+\dots$

$=$

$=-25x+\frac{625x^3}{3!}-\frac{5^6}{5!}x^5+\dots$

$=-5\left(5x-\frac{125}{3!}x^3+\frac{5^5}{5!}x^5+\dots\right)=-5(\sin 5x)$ ✓